Chapter 5

The Time Value of Money

Learning Objectives

1. Explain the mechanics of compounding, and bringing the value of money back to the present.
2. Understand annuities.
3. Determine the future or present value of a sum when there are nonannual compounding periods.
4. Determine the present value of an uneven stream of payments and understand perpetuities.

COMPOUND INTEREST, FUTURE, AND PRESENT VALUE

Using Timelines to Visualize Cash Flows

Timeline of cash flows

$r = 10\%$

YEARS	0	1	2	3	4
Cash Flow	−100	30	20	−10	50
	Today & Beginning of Period 1		End of Period 2 & Beginning of Period 3		End of Period 4 & Beginning of Period 5

Simple Interest

- Interest is earned only on principal.
- Example: Compute simple interest on $100 invested at 6% per year for three years.

1st year	interest is $6.00
2nd year	interest is $6.00
3rd year	interest is $6.00

Total interest earned: **$18.00**

Compound Interest

- Compounding is when interest paid on an investment during the first period is added to the principal; then, during the second period, interest is earned on the new sum (that includes the principal <u>and</u> interest earned so far).

Compound Interest

- Example: Compute compound interest on $100 invested at 6% for three years with annual compounding.

  1st year interest is $6.00 Principal now is $106.00

  2nd year interest is $6.36 Principal now is $112.36

  3rd year interest is $6.74 Principal now is $119.10

  Total interest earned: **$19.10**

Future Value

- Future Value is the amount a sum will grow to in a certain number of years when compounded at a specific rate.

$$FVN = PV\,(1 + r)^n$$

- FVN = the future of the investment at the end of "n" years
- r = the annual interest (or discount) rate
- n = number of years
- PV = the present value, or original amount invested at the beginning of the first year

Future Value Example

- Example: What will be the *FV* of \$100 in 2 years at interest rate of 6%?

$$FV_2 = PV(1 + r)^2 = \$100\ (1 + 0.06)^2$$
$$= \$100\ (1.06)^2$$
$$= \mathbf{\$112.36}$$

How to Increase the Future Value?

- Future Value can be increased by:
  - Increasing number of years of compounding (*N*)
  - Increasing the interest or discount rate (*r*)
  - Increasing the original investment (*PV*)
- See example on next slide

Changing *R*, *N*, and *PV*

a. You deposit $500 in bank for 2 years. What is the *FV* at 2%? What is the *FV* if you change interest rate to 6%?

   FV at 2% = $500*(1.02)^2$ = $520.20

   FV at 6% = $500*(1.06)^2$ = $561.80

b. Continue the same example but change time to 10 years. What is the *FV* now?

   FV = $500*(1.06)^{10}$= $895.42

c. Continue the same example but change contribution to $1,500. What is the *FV* now?

   FV = $1,500*(1.06)^{10}$ = $2,686.27

FIGURE 5-1 $100 Compounded at 6 Percent over 20 Years

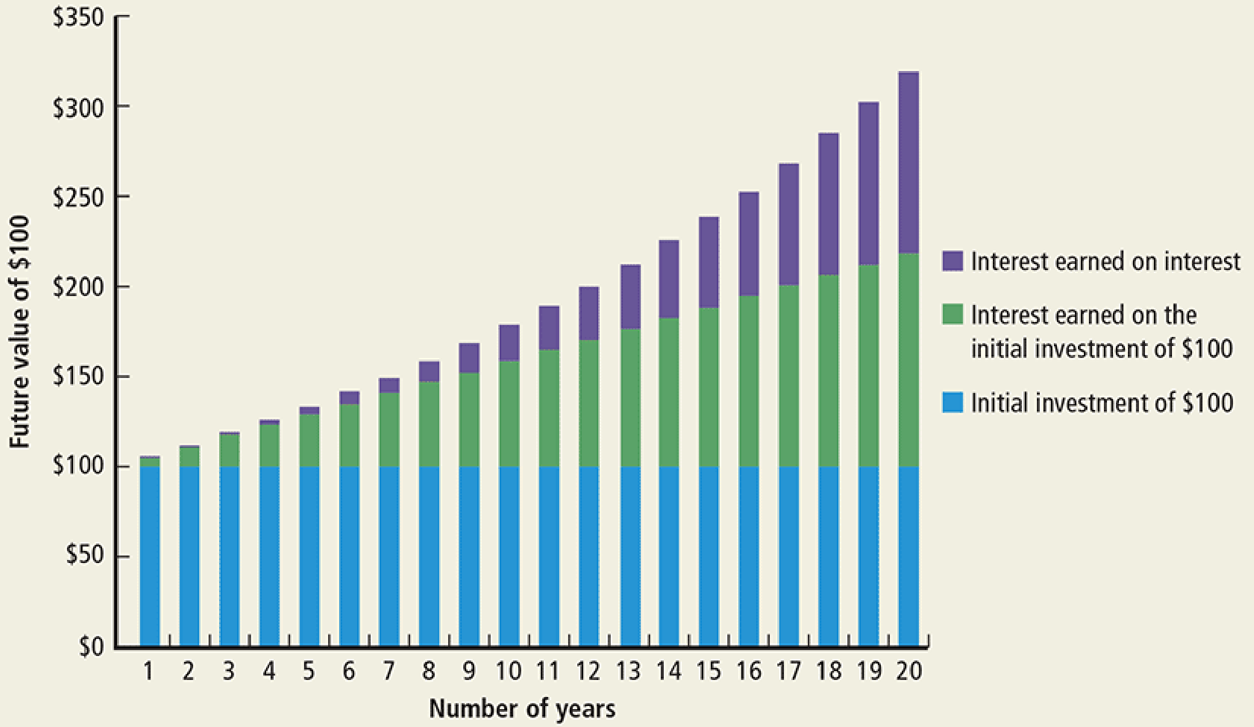

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FIGURE 5-2 The Future Value of $100 Initially Deposited and Compounded at 0, 5, 10, and 15 Percent

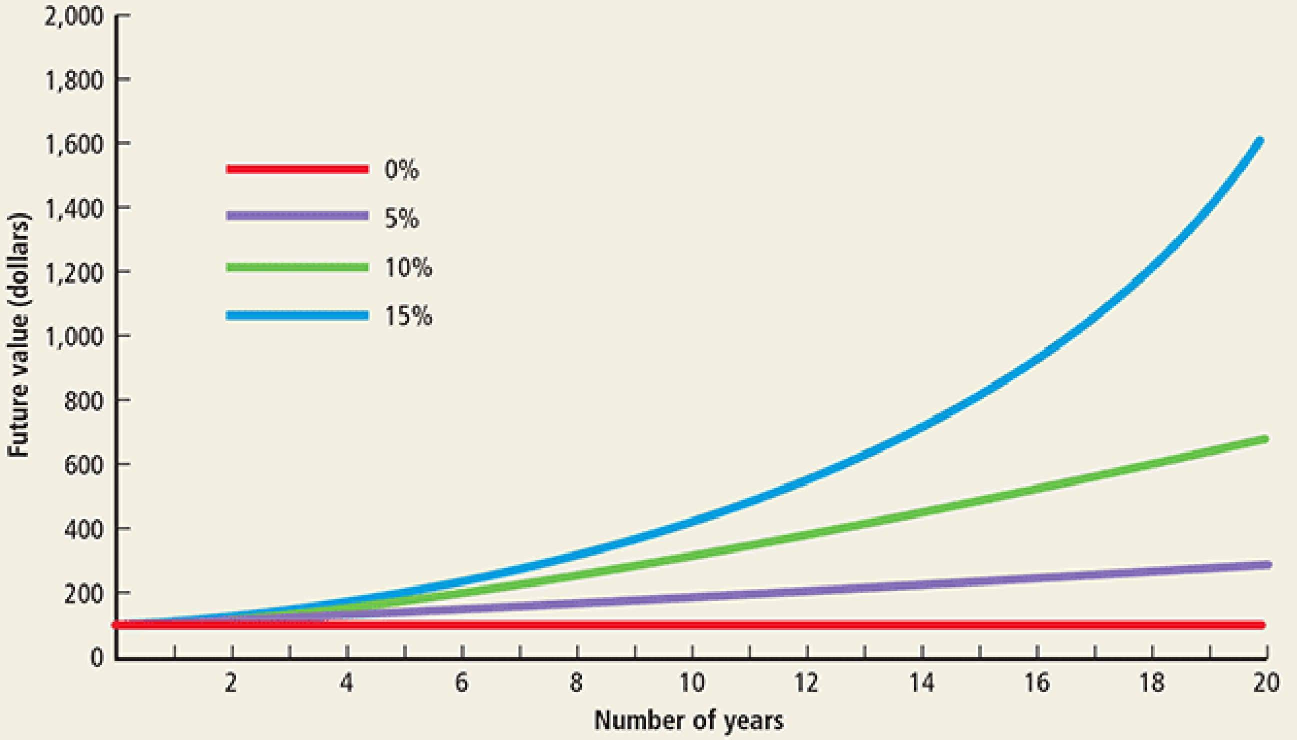

Figure 5-2

- Figure 5-2 illustrates that we can increase the *FV* by:
  - Increasing the number of years for which money is invested; and/or
  - Investing at a higher interest rate.

Computing Future Values using Calculator or Excel

- Review discussion in the text book

- Excel Function for *FV*:
  - = FV(rate,nper,pmt,pv)

Present Value

- Present value reflects the current value of a future payment or receipt.

Present Value

$$PV = FVn\ \{1/(1 + r)^n\}$$

FVn = the future value of the investment at the end of n years

n = number of years until payment is received

r = the interest rate

PV = the present value of the future sum of money

PV example

- What will be the present value of $500 to be received 10 years from today if the discount rate is 6%?
- $PV = \$500\ \{1/(1+0.06)^{10}\}$

  $= \$500\ (1/1.791)$

  $= \$500\ (0.558)$

  = **$279.00**

FIGURE 5-3 The Present Value of $100 to Be Received at a Future Date and Discounted Back to the Present at 0, 5, 10, and 15 Percent

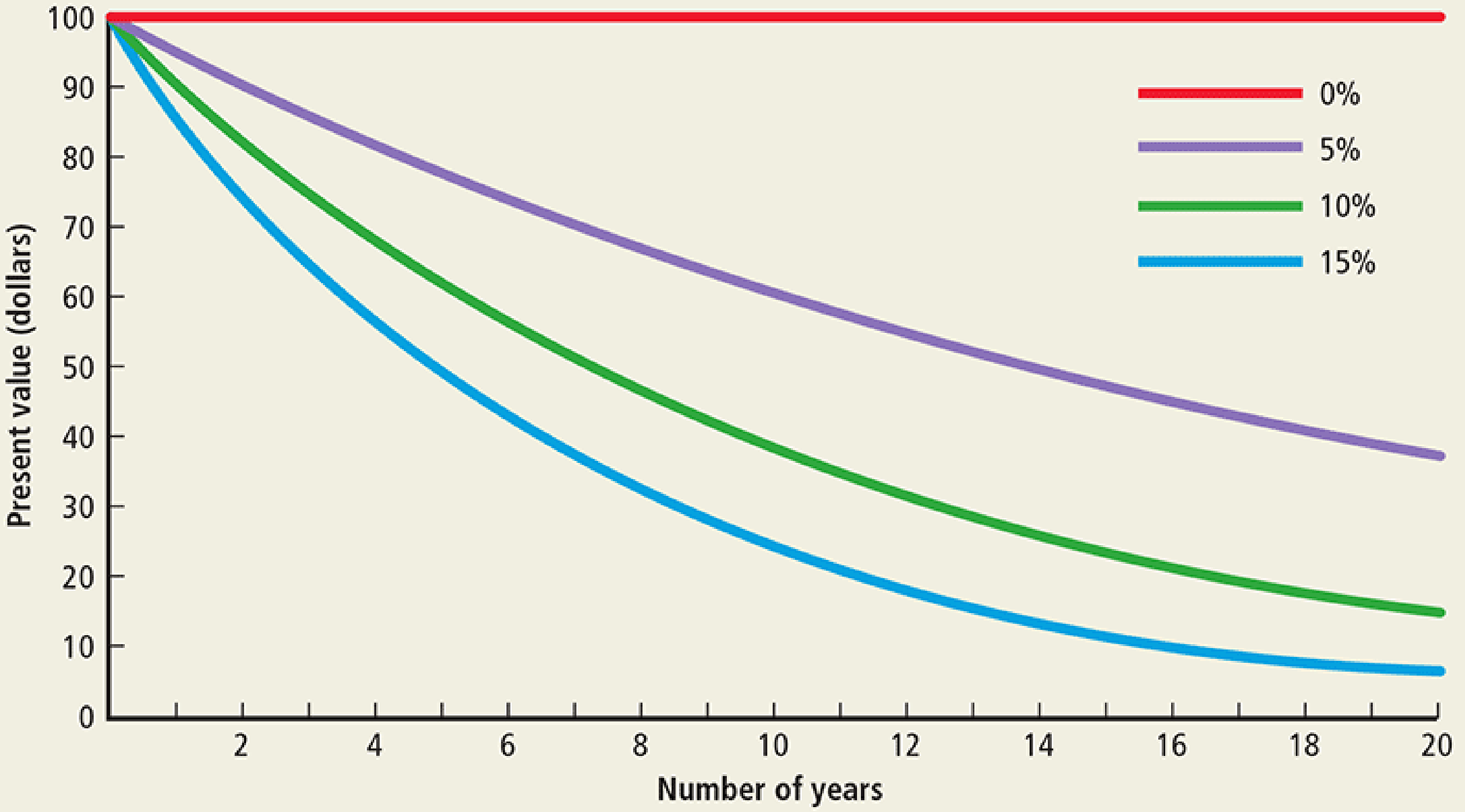

Figure 5-3

- Figure 5-3 illustrates that *PV* is lower if:
  - Time period is longer; and/or
  - Interest rate is higher.

Using Excel

- Excel Function for *PV:*
  - = PV(rate,nper,pmt,fv)

ANNUITIES

Annuity

- An annuity is a series of equal dollar payments for a specified number of years.

- Ordinary annuity payments occur at the end of each period.

FV of Annuity

Compound Annuity

- Depositing or investing an equal sum of money at the end of each year for a certain number of years and allowing it to grow.

FV Annuity - Example

- What will be the *FV* of a 5-year, $500 annuity compounded at 6%?
- FV_5 = $500 $(1 + 0.06)^4$ + $500 $(1 + 0.06)^3$
  + 500(1 + 0.06)^2$ + $500 (1 + 0.06) + $500
  = $500 (1.262) + $500 (1.191) + $500 (1.124)
  + $500 (1.090) + $500
  = $631.00 + $595.50 + $562.00 + $530.00 + $500
  = **$2,818.50**

TABLE 5-1 Growth of a 5-Year, $500 Annuity Compounded at 6 Percent

$r = 6\%$

YEAR	0	1	2	3	4	5
Dollar deposits at end of year		500	500	500	500	500
						$ 500.00
						530.00
						562.00
						595.50
						631.00
Future value of the annuity						$2,818.50

FV of an Annuity – Using the Mathematical Formulas

$$FV_n = PMT \{(1 + r)^n - 1/r\}$$

FV_n = the future of an annuity at the end of the nth year

PMT = the annuity payment deposited or received at the end of each year

r = the annual interest (or discount) rate

n = the number of years

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FV of an Annuity – Using the Mathematical Formulas

- What will $500 deposited in the bank every year for 5 years at 6% be worth?

- $FV = PMT\ ([(1 + r)^n - 1]/r)$

  $= \$500\ (5.637)$

  $= \mathbf{\$2,818.50}$

FV of Annuity: Changing *PMT*, *N*, and *r*

1. What will $5,000 deposited annually for 50 years be worth at 7%?

   FV = $2,032,644

   Contribution = $250,000 (= 5000*50)

2. <u>Change *PMT*</u> = $6,000 for 50 years at 7%

   FV = $2,439,173

   Contribution= $300,000 (= 6000*50)

FV of Annuity: Changing *PMT*, *N*, and *r*

3. Change time = 60 years, $6,000 at 7%

   FV = $4,881,122

   Contribution = $360,000 (= 6000*60)

4. Change *r* = 9%, 60 years, $6,000

   FV = $11,668,753

   Contribution = $360,000 (= 6000*60)

Present Value of an Annuity

- Pensions, insurance obligations, and interest owed on bonds are all annuities. To compare these three types of investments we need to know the present value (*PV*) of each.

TABLE 5-2 Illustration of a 5-Year, $500 Annuity Discounted to the Present at 6 Percent

r = 6%

YEAR	0	1	2	3	4	5
Dollars received at end of year		500	500	500	500	500
	$ 471.50	←				
	445.00	←	←			
	420.00	←	←	←		
	396.00	←	←	←	←	
	373.50	←	←	←	←	←
Present value of the annuity	$2,106.00					

PV of Annuity – Using the Mathematical Formulas

- *PV* of Annuity $= PMT \{[1 - (1 + r)^{-1}]\}/r$
  $= 500\ (4.212)$
  $=$ **\$2,106**

Annuities Due

- Annuities due are ordinary annuities in which all payments have been shifted forward by one time period. Thus, with annuity due, each annuity payment occurs at the beginning of the period rather than at the end of the period.

Annuities Due

- Continuing the same example: If we assume that $500 invested every year for 5 years at 6% to be annuity due, the future value will increase due to compounding for one additional year.
- FV_5 (annuity due) = $PMT\ \{[(1 + r)^n - 1]/r\}\ (1 + r)$

$$= 500(5.637)(1.06)$$

$$= \mathbf{\$2{,}987.61}$$

(versus $2,818.80 for ordinary annuity)

Amortized Loans

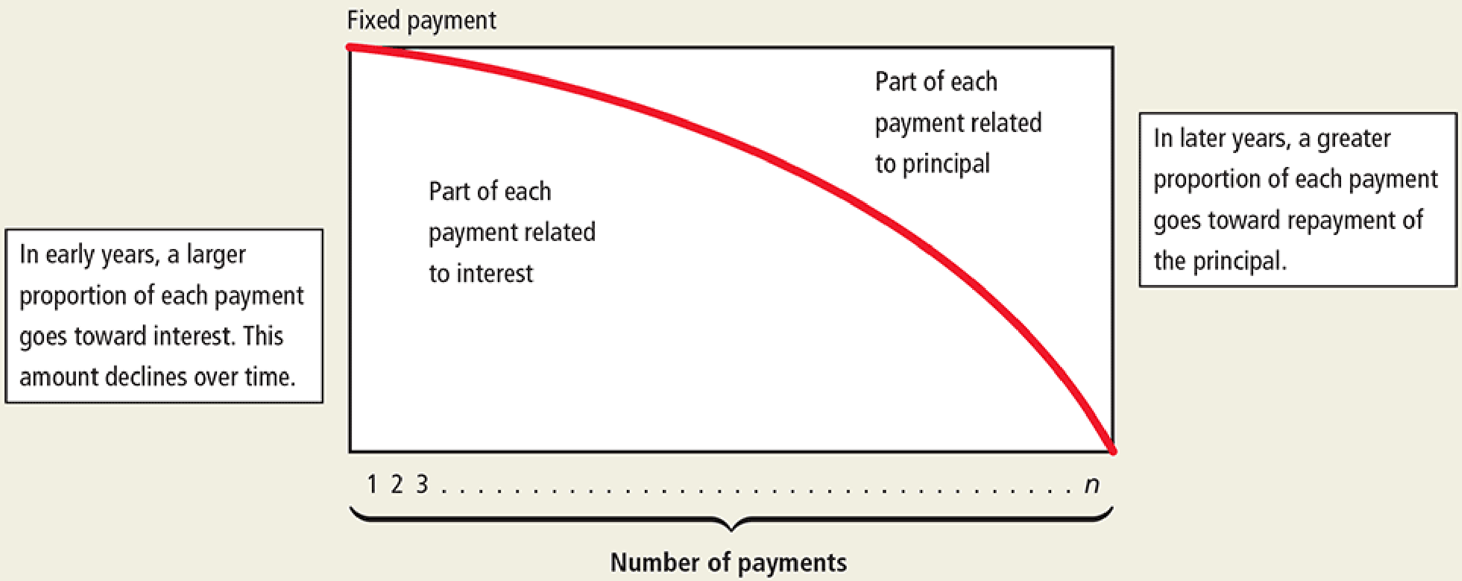
- Loans paid off in equal installments over time are called amortized loans. *Example*: Home mortgages, auto loans.

- Reducing the balance of a loan via annuity payments is called amortizing.

Amortized Loans

- The periodic payment is fixed. However, different amounts of each payment are applied toward the principal and interest. With each payment, you owe less toward principal. As a result, the amount that goes toward interest declines with every payment (as seen in Figure 5-4).

FIGURE 5-4 The Amortization Process



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Amortization Example

- *Example*: If you want to finance a new machinery with a purchase price of $6,000 at an interest rate of 15% over 4 years, what will your annual payments be?

Finding *PMT* – Using the Mathematical Formulas

- Finding Payment: Payment amount can be found by solving for *PMT* using *PV* of annuity formula.
- *PV* of Annuity = $PMT\ \{1 - (1 + r)^{-4}\}/r$

$$6{,}000 = PMT\ \{1 - (1 + 0.15)^{-4}\}/0.15$$

$$6{,}000 = PMT\ (2.855)$$

$$PMT = 6{,}000/2.855$$

$$= \mathbf{\$2{,}101.59}$$

TABLE 5-3 Loan Amortization Schedule Involving a $6,000 Loan at 15 Percent to Be Repaid in 4 Years

Year	Annuity	Interest Portion of the Annuity[a]	Repayment of the Principal Portion of the Annuity[b]	Outstanding Loan Balance After the Annuity Payment
0	0	0	0	$6,000.00
1	$2,101.59	$900.00	$1,201.59	4,798.41
2	2,101.59	719.76	1,381.83	3,416.58
3	2,101.59	512.49	1,589.10	1,827.48
4	2,101.59	274.12	1,827.48	

[a]The interest portion of the annuity is calculated by multiplying the outstanding loan balance at the beginning of the year by the interest rate of 15 percent. Thus, for year 1 it was $\$6{,}000 \times 0.15 = \$900.00$, for year 2 it was $\$4{,}798.41 \times 0.15 = \$719.76$, and so on.

[b]Repayment of the principal portion of the annuity was calculated by subtracting the interest portion of the annuity (column 2) from the annuity (column 1).

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MAKING INTEREST RATES COMPARABLE

Making Interest Rates Comparable

- We cannot compare rates with different compounding periods. For example, 5% compounded annually is not the same as 5% percent compounded quarterly.
- To make the rates comparable, we compute the annual percentage yield (APY) or effective annual rate (EAR).

Quoted Rate versus Effective Rate

- Quoted rate could be very different from the effective rate if compounding is not done annually.
- *Example*: $1 invested at 1% per month will grow to $1.126825 (= $\$1.00(1.01)^{12}$) in one year. Thus even though the interest rate may be quoted as 12% compounded monthly, the effective annual rate or APY is 12.68%.

Quoted Rate versus Effective Rate

- $APY = (1 + \text{quoted rate}/m)^m - 1$

  Where m = number of compounding periods

  $= (1 + 0.12/12)^{12} - 1$

  $= (1.01)^{12} - 1$

  = **.126825** or 12.6825%

TABLE 5-4 The Value of $100 Compounded at Various Intervals

For 1 Year At *r* Percent				
r =	2%	5%	10%	15%
Compounded annually	$102.00	$105.00	$110.00	$115.00
Compounded semiannually	102.01	105.06	110.25	115.56
Compounded quarterly	102.02	105.09	110.38	115.87
Compounded monthly	102.02	105.12	110.47	116.08
Compounded weekly (52)	102.02	105.12	110.51	116.16
Compounded daily (365)	102.02	105.13	110.52	116.18
For 10 Years At *r* Percent				
r =	2%	5%	10%	15%
Compounded annually	$121.90	$162.89	$259.37	$404.56
Compounded semiannually	122.02	163.86	265.33	424.79
Compounded quarterly	122.08	164.36	268.51	436.04
Compounded monthly	122.10	164.70	270.70	444.02
Compounded weekly (52)	122.14	164.83	271.57	447.20
Compounded daily (365)	122.14	164.87	271.79	448.03

Finding *PV* and *FV* with Nonannual Periods

- If interest is not paid annually, we need to change the interest rate and time period to reflect the nonannual periods while computing *PV* and *FV*.

  r = stated rate/# of compounding periods

  N = # of years * # of compounding periods in a year

- <u>Example</u>: If your investment earns 10% a year, with quarterly compounding for 10 years, what should we use for "*r*" and "*N*" ?

  r = 0.10/4 = 0.025 or 2.5%

  N = 10*4 = 40 periods

THE PRESENT VALUE OF AN UNEVEN STREAM AND PERPETUITIES

The Present Value of an Uneven Stream

- Some cash flow stream may not follow a conventional pattern. For example, the cash flows may be erratic (with some positive cash flows and some negative cash flows) or cash flows may be a combination of single cash flows and annuity (as illustrated in Table 5-5).

TABLE 5-5 The Present Value of an Uneven Stream Involving One Annuity Discounted to the Present at 6 Percent: An Example

$r = 6\%$

YEAR	0	1	2	3	4	5	6	7	8	9	10
Dollars received at end of year		0	200	−400	500	500	500	500	500	500	500

	$ 178.00 (← year 2)
	−335.85 (← year 3)
	$2,791.19 (← years 4–10)
	2,343.54 (← $2,791.19)
Total present value	$2,185.69

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TABLE 5-6 Determining the Present Value of an Uneven Stream Involving One Annuity Discounted to the Present at 6 Percent: An Example

1. Present value of $200 received at the end of 2 years at 6% =	$ 178.00
2. Present value of a $400 outflow at the end of 3 years at 6% =	−335.85
3. (a) Value at end of year 3 of a $500 annuity, years 4–10 at 6% = $2,791.19 (b) Present value of $2,791.19 received at the end of year 3 at 6% =	2,343.54
4. Total present value =	$2,185.69

Perpetuity

- A perpetuity is an annuity that continues forever.
- The present value of a perpetuity is given by ***PV = PP/r***
- *PV* = present value of the perpetuity
- *PP* = constant dollar amount provided by the perpetuity
- *r* = annual interest (or discount) rate

Perpetuity

- *Example*: What is the present value of $2,000 perpetuity discounted back to the present at 10% interest rate?

= 2000/0.10

= **$20,000**

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TABLE 5-7 Summary of Time Value of Money Equations

Calculation	Equation
Future value of a single payment	$FV_n = PV(1 + r)^n$
Present value of a single payment	$PV = FV_n\left[\frac{1}{(1 + r)^n}\right]$
Future value of an annuity	$FV \text{ of an annuity} = PMT\left[\frac{(1 + r)^n - 1}{r}\right]$
Present value of an annuity	$PV \text{ of an annuity} = PMT\left[\frac{1 - (1 + r)^{-n}}{r}\right]$
Future value of an annuity due	$FV_n(\text{annuity due}) = \text{future value of an annuity} \times (1 + r)$
Present value of an annuity due	$PV(\text{annuity due}) = \text{present value of an annuity} \times (1 + r)$
Effective annual rate (EAR)	$EAR = \left(1 + \frac{\text{APR or quoted annual rate}}{\text{compounding periods per year } (m)}\right)^m - 1$
Future value of a single payment with nonannual compounding	$FV_n = PV\left[1 + \frac{APR}{m}\right]^{m \cdot n}$
Present value of a perpetuity	$PV = \frac{PP}{r}$

Notations:
FV_n = the future value of the investment at the end of n years
n = the number of years until payment will be received or during which compounding occurs
r = the annual interest or discount rate
PV = the present value of the future sum of money
m = the number of times compounding occurs during the year
PMT = the annuity payment deposited or received at the end of each year
PP = the constant dollar amount provided by the perpetuity

Key Terms

- Amortized loan
- Annuity
- Annuity due
- Annuity future value factor
- Annuity present value factor
- Compound annuity
- Compound interest
- Effective annual rate (EAR)
- Future value
- Future value factor
- Ordinary annuity
- Present value
- Present value factor
- Perpetuity
- Simple interest